MATH 140A Review: Sequences and Series

Facts to Know:

How do we add an infinite list of numbers?

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

Answer:

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$S_N = a_1 + a_2 + a_3 + \cdots + a_N = \sum_{n=1}^{N} a_n$$

If the limit of the sequence $\{S_N\}_{N=1}^{\infty}$ exists, then define

$$\sum_{n=1}^{\infty} a_n = \lim_{N \to \infty} S_N = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$$

and say that the series converges. If the limit div., then we say that the series diverges.

Example: Determine if the series

$$\sum_{n=9}^{\infty} \ln \frac{1+\frac{1}{n}}{1+\frac{1}{n-1}}$$

converges? If it converges, what does the series add up to?

Solution. We have that

$$S_N = \sum_{n=9}^{N} \ln\left(\frac{1+\frac{1}{n}}{1+\frac{1}{n-1}}\right)$$
$$= \sum_{n=9}^{N} \left(\ln\left(1+\frac{1}{n}\right) - \ln\left(1+\frac{1}{n-1}\right)\right)$$
$$= \left(\cancel{\ln\left(1+\tfrac{1}{9}\right)} - \ln\left(1+\tfrac{1}{8}\right)\right)$$
$$+ \left(\cancel{\ln\left(1+\tfrac{1}{10}\right)} - \cancel{\ln\left(1+\tfrac{1}{9}\right)}\right)$$
$$+ \left(\cancel{\ln\left(1+\tfrac{1}{11}\right)} - \cancel{\ln\left(1+\tfrac{1}{10}\right)}\right)$$
$$\vdots$$
$$+ \left(\ln\left(1+\tfrac{1}{N}\right) - \cancel{\ln\left(1+\tfrac{1}{N-1}\right)}\right)$$

$$= \left(-\ln\left(1+\frac{1}{8}\right)\right) + \ln\left(1+\frac{1}{N}\right).$$

Thus,

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(-\ln\left(1+\frac{1}{8}\right) + \ln\left(1+\frac{1}{N}\right)\right)$$

$$= -\ln\left(1+\frac{1}{8}\right) + 0.$$

The series conv. and adds up to $-\ln\left(1+\frac{1}{8}\right)$.

$\square$

Facts to Know:

Let $r \in \mathbb{R}$. The sequence $a_n = r^n$ is called the *geometric sequence*.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & |r|<1 \\ 1 & r=1 \\ \text{DNE} & r \le -1 \\ \infty & 1<r \end{cases}$$

The *geometric series*

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r},$$

converges for $|r|<1$ and diverges otherwise.

Example: Determine if the following series converges. If so, what is the sum?

$$\sum_{n=2}^{\infty} 3 \cdot \frac{1}{4^n}.$$

Solution. We have

$$\sum_{n=2}^{\infty} 3 \cdot \frac{1}{4^n} = 3 \cdot \sum_{n=2}^{\infty} \left(\frac{1}{4}\right)^n$$

$$= 3\left(\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{n+2}\right)$$

$$= 3 \cdot \left(\frac{1}{4}\right)^2 \left(\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n\right)$$

$$= 3 \cdot \left(\frac{1}{4}\right)^2 \cdot \left(\frac{1}{1-\frac{1}{4}}\right),$$

where the last holds by using geometric series for $r=\frac{1}{4}$ and since $\left|\frac{1}{4}\right|<1$. Thus, the series converges.

$\square$